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ON THE SPAN OF A RANDOM CHANNEL ASSIGNMENT PROBLEM

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In the radio channel assignment problems considered here, we must assign a ‘channel’ from the set $1,2,\dots$ of positive integers to each of n transmitters, and we wish to minimise the span of channels used, subject to the assignment leading to an acceptable level of interference. A standard form of this problem is the ‘constraint matrix’ model. The simplest case of this model (the $0,1$ case) is essentially graph colouring. We consider here a random model for the next simplest case (with lengths 0, 1 or 2), and determine the asymptotic behaviour of the span of channels needed as $n \to \infty$. We find that there is a ‘phase change’ in this behaviour, depending on the probabilities for the different lengths.

1. Introduction

A standard model for radio channel assignment is the *constraint matrix* or *weighted graph* model (with unit demands). We are given a set V of n transmitters, to each of which we must assign a ‘channel’ from $1,2,\dots,t$. There is a *constraint graph* $G=(V,E)$ on these nodes, together with a positive integer length x_{uv} for each edge uv, which specifies the ‘minimum allowed channel separation’. (We shall refer to the integers x_{uv} as lengths rather than weights.) Thus an assignment $\phi : V \to \{1,\dots,t\}$ is *feasible* if for each pair of distinct transmitters u and v we have $|\phi(u)-\phi(v)| \geq x_{uv}$. The *span* of the problem, $\text{span}(G,x)$, is the least t for which there is a feasible assignment ϕ.

It is also possible to think of the problem as being specified by the complete graph on the set V, together with a non-negative integer x_e on each

Mathematics Subject Classification (2000): 05C80, 90B15, 05C15 edge e; or by an $(n \times n)$ symmetric matrix of non-negative integers, the 'constraint matrix'. (The diagonal entries specify the 'co-site' constraints, which are not relevant here since we consider only the unit demand case.)

We consider the case where the lengths (the minimum allowed channel separations) x_e may take only the values 1 or 2. This is the simplest case that takes us beyond graph colouring. Let E_i denote the set of edges of length i. Then E_2 contains the 'long' edges, E_1 the 'short' edges, and E_0 the 'missing' edges.

Recall that $\chi(G)$ denotes the *chromatic number* of the graph G, which is the least number of colours in a colouring of the nodes such that no two adjacent nodes get the same colour. If each edge length is 1, then we are back to ordinary graph colouring, and $\text{span}(G, x) = \chi(G)$. Since we always insist here that each edge length is 1 or 2, we have

$$\chi(G) \leq \text{span}(G, x) \leq 2\chi(G) - 1. \tag{1}$$

The upper bound in (1) is easy if we use only the odd channels, since each $x_e \leq 2$. Our results will fall into two cases, when there are 'few long edges' and when there are 'few short edges'. We already noted that if there are no long edges then $\text{span}(G, x) = \chi(G)$; if there are no short edges then $\text{span}(G, x) = 2\chi(G) - 1$, see the comments in the next section.

Let us introduce the random model. First let us recall the standard random graph $G_{n,p}$, see for example [1,3]. Given $0 \leq p \leq 1$ and a positive integer n, the random graph $G_{n,p}$ has nodes $v_1, \ldots, v_n$ and the $\binom{n}{2}$ possible edges appear independently, each with probability p.

Now let p_0, p_1 and p_2 be non-negative and sum to 1, and let $\mathbf{p} = (p_0, p_1, p_2)$. We call $\mathbf{p}$ a *probability vector*. The *random network* $G_{n,\mathbf{p}}$ has nodes $v_1, \ldots, v_n$ and the $\binom{n}{2}$ edges e have independent lengths X_e, where $\Pr(X_e = i) = p_i$. An edge of length 0 corresponds to a missing edge, so the constraint graph associated with the network has distribution G_{n,p_1+p_2}.

It is well known [2] that

$$\chi(G_{n,p}) \sim \frac{1}{2} \ln\left(\frac{1}{1-p}\right) \frac{n}{\ln n}. \tag{2}$$

(We take p as fixed.) This notation means that the ratio of left hand side to right hand side tends to 1 in probability. (Much more precise results are known, see for example [6].) At the recent Workshop on Radio Channel Assignment in Brunel University in July 2000, Jan van den Heuvel asked if there were similar results for the asymptotic behaviour of $\text{span}(G_{n,\mathbf{p}})$. This paper is devoted to answering his question.

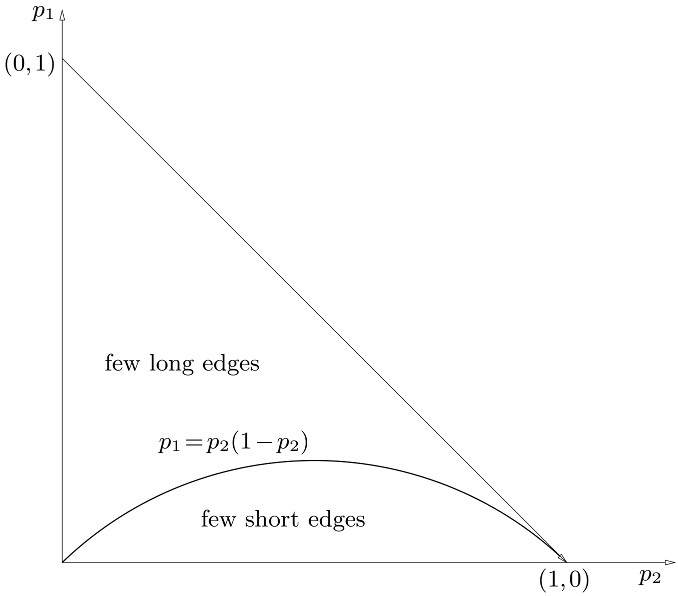

Figure 1. The two regions: projection on the p_0-plane

Fix a probability vector $\mathbf{p} = (p_0, p_1, p_2)$. It turns out that there is an abrupt change or 'phase transition' in the behaviour of $\mathrm{span}(G_{n,\mathbf{p}})$ as we cross the curve $p_1 = p_2(1-p_2)$ in the projection of the probability vectors on the p_0-plane – see Figure 1. The 'critical points' are those of the form $(p_0, p_0^{\frac{1}{2}}(1-p_0^{\frac{1}{2}}), 1-p_0^{\frac{1}{2}})$. If $p_1 \le p_2(1-p_2)$ we are in the 'few short edges' regime, where we may as well treat short edges as long and leave about half the channel sets empty. (A *channel set* is the set of nodes assigned a given channel.) In contrast, if $p_1 \ge p_2(1-p_2)$ we are in the 'few long edges' regime, and it turns out that it is best to choose the channel sets nearly uniform in size.

Theorem 1.1. *Consider a fixed probability vector* $\mathbf{p} = (p_0, p_1, p_2)$, *where* $p_0, p_1, p_2 > 0$ *and* $p_0 + p_1 + p_2 = 1$. *If* $p_1 \le p_2(1-p_2)$ *then*

$$\mathrm{span}(G_{n,\mathbf{p}}) \sim 2\chi(G_{n,p_1+p_2}) \sim \ln\left(\frac{1}{p_0}\right)\frac{n}{\ln n}; \tag{3}$$

and if $p_1 \ge p_2(1-p_2)$ *then*

$$\mathrm{span}(G_{n,\mathbf{p}}) \sim \left(\frac{1}{2}\ln\left(\frac{1}{p_0}\right) + \ln\left(\frac{1}{1-p_2}\right)\right)\frac{n}{\ln n}. \tag{4}$$

On the ‘critical curve’ $p_1 = p_2(1-p_2)$ we have $\ln(1/p_0) = 2\ln(1/(1-p_2))$, so the two expressions in the theorem for $\mathrm{span}(G_{n,\mathbf{p}})$ are equal there. It follows from the theorem together with (2), that when $p_1 \geq p_2(1-p_2)$ we have

$$\mathrm{span}(G_{n,\mathbf{p}}) \sim \chi(G_{n,p_1+p_2}) + 2\chi(G_{n,p_2}),$$

but it is not clear what to make of this. The behaviour of $\mathrm{span}(G_{n,\mathbf{p}})$ when some $p_i = 0$ is discussed at the end of the next section.

Given a probability vector $\mathbf{p}$ with each $p_i > 0$, let

$$\gamma(\mathbf{p}) = \min\left\{-\ln p_0, -\frac{1}{2}\ln p_0 - \ln(1-p_2)\right\}.$$

Thus $\gamma(\mathbf{p}) = -\ln p_0$ if $p_1 \leq p_2(1-p_2)$ (the ‘few short edges’ case) and $\gamma(\mathbf{p}) = -\frac{1}{2}\ln p_0 - \ln(1-p_2)$ otherwise. Then the theorem above says that

$$\mathrm{span}(G_{n,\mathbf{p}}) \sim \gamma(\mathbf{p})\ n/\ln n.$$

Note that $\gamma(\mathbf{p})$ is not differentiable at the critical points.

The bounded differences approach immediately yields a concentration result for the span, just as it does for the chromatic number $\chi(G_{n,p})$. Note that changing the edge lengths incident with a node can change the span by at most 2. Hence, see for example [7,8], for any $t > 0$

$$\Pr(|\mathrm{span}(G_{n,\mathbf{p}}) - \mathrm{E}[\mathrm{span}(G_{n,\mathbf{p}})]| \geq t) \leq 2e^{-\frac{t^2}{2n}}.$$

By the last result and the theorem, for any fixed $\varepsilon > 0$

(5) $$\Pr\left(\left|\frac{\mathrm{span}(G_{n,\mathbf{p}})}{\gamma(\mathbf{p})n/\ln n} - 1\right| \geq \varepsilon\right) = e^{-\Omega(n/\ln^2 n)}.$$

A natural lower bound for the chromatic number $\chi(G)$ is the clique number $\omega(G)$. For fixed p with $0 < p < 1$, we have $\omega(G_{n,p}) \sim 2\ln n/\ln(1/p)$, see for example [1,3]. Thus we see from (2) that this lower bound is hopelessly weak for $\chi(G_{n,p})$. Similarly, clique-based lower bounds for the span (see for example [9]) are hopelessly weak here. A very different approach to modelling random channel assignment problems is taken in [10], where it is assumed that transmitters are scattered at random in the plane. In that paper, a focus is to compare the chromatic number or span to clique-based lower bounds.

[We shall obtain tight lower bounds for the span from an extension of another standard lower bound on the chromatic number $\chi(G)$, namely that $\chi(G) \geq |V|/\alpha(G)$, where the *stability number* $\alpha(G)$ is the maximum size of a set of nodes no two of which are adjacent – see (10) below.]

2. Some extreme cases

Let us amplify the comments above about the deterministic inequality (1), see also [9]. After that we are able to deal easily with the behaviour of $\text{span}(G_{n,\mathbf{p}})$ as in the theorem above but when some $p_i = 0$. We consider three cases.

(a) *Few long edges.* We noted that if E_2 is empty then $\text{span}(G,x) = \chi(G)$. Indeed, that result is all we shall need here, but we can extend it if E_2 is sufficiently small. Write χ for $\chi(G)$. Then $\text{span}(G,x) = \chi$ if $|E_2| \le \chi - 2$, and always $\text{span}(G,x) \le \chi + (2/\chi)|E_2|$. To see the first result, colour the nodes of G with χ colours, and consider the graph $\tilde{G}$ on $t = \chi$ nodes formed by 'contracting' the colour sets to single nodes, where two nodes of $\tilde{G}$ are adjacent when they contain adjacent nodes from G. Call an edge of $\tilde{G}$ *bad* if at least one of the corresponding edges of G was long. If $|E_2| \le \chi - 2$ then $\tilde{G}$ has at most $\chi - 2$ bad edges. But a complete graph on t nodes with at most $t-2$ 'bad' edges has a Hamilton path not using these bad edges ([11], see for example [13], exercise 6.2.25), and we can use such a path to assign a channel to each colour set, as for example in [12].
The second result follows in a similar manner, since a complete graph on t nodes with r bad edges has a Hamilton path using at most $(2/t)r$ bad edges. (To see this, pick a Hamilton path uniformly at random.)

(b) *Few short edges.* If E_1 is empty, then $\text{span}(G,x) = 2\chi(G) - 1$, as has been noted many times, see for example [9]. We have already seen the upper bound. For the lower bound, observe that in any feasible assignment which uses an even channel, we can always push the lowest such channel down by 1. Hence there is a feasible assignment using just odd integers in $1,\ldots,\text{span}(G,x)$, and so $\chi(G) \le \frac{1}{2}(\text{span}(G,x)+1)$. This proof extends to show that, for any integral length vector x and positive integer k, we have

$$\text{span}(G,kx) = 1 + k(\text{span}(G,x) - 1).$$

(c) *No missing edges.* If E_0 is empty, that is if the graph G is complete, then $\text{span}(G,x)$ is $1+$ the minimum length of a Hamilton path, as in (a).

Let us finish this section by considering the behaviour of $\text{span}(G_{n,\mathbf{p}})$ when some $p_i = 0$. The above comments allow us to handle this task quickly. Note first that if some $p_i = 1$ then things are trivial – almost surely (a.s.) the span is 1 if $p_0 = 1$, n if $p_1 = 1$ and $2n-1$ if $p_2 = 1$. So assume now that this is not the case. If $p_2 = 0$ then by (a) above, a.s. $\text{span}(G_{n,\mathbf{p}}) = \chi(G_{n,p_1})$. If $p_1 = 0$ then by (b) above, a.s. $\text{span}(G_{n,\mathbf{p}}) = 2\chi(G_{n,p_2}) - 1$. In both these cases then the behaviour is well understood – see (2). Finally suppose that $p_0=0$. Since $p_1>0$ the random graph G_{n,p_1} has a Hamilton path asymptotically almost surely (a.a.s.), that is, with probability tending to 1 as $n\to\infty$ – see for example [1] or [3]. Hence it follows by (c) above that $\mathrm{span}(G_{n,\mathbf{p}})=n$ a.a.s.

3. Starting the proof

We have already discussed the case when at least one of p_0, p_1 and p_2 is 0, so let us assume throughout from now on that each p_i is strictly positive, as in the theorem.

Let n and t be positive integers. Let U be a (fixed) non-empty set of nodes in $G_{n,\mathbf{p}}$, and let $\phi\colon U\to\{1,\dots,t\}$. We need bounds on the probability that ϕ is feasible (for the subnetwork induced by U). Let $n_i=|\phi^{-1}(i)|$ for $i=1,\dots,t$, so that $\sum_{i=1}^t n_i=|U|$. Then

$$\Pr(\phi \text{ feasible}) = p_0^{\sum_{i=1}^t \binom{n_i}{2}}(1-p_2)^{\sum_{i=1}^{t-1} n_i n_{i+1}}.$$

Hence

$$\ln \Pr(\phi \text{ feasible}) = -a\sum_{i=1}^t \binom{n_i}{2} - b\sum_{i=1}^{t-1} n_i n_{i+1}, \tag{6}$$

where $a=\ln\frac{1}{p_0}$ and $b=\ln\frac{1}{1-p_2}$. Note that $a>b>0$. We need to consider quantities like that on the right hand side of (6) above, and that is the subject of the next section.

4. Extremal configurations

This section contains technical results concerning sums like those that appear in the equation (6) above. The first three lemmas are needed later: the remainder of the section is devoted to their proof.

In this section, let a and b be any positive constants. After that, we shall revert to setting $a=\ln\frac{1}{p_0}$ and $b=\ln\frac{1}{1-p_2}$. Always t will be a positive integer and $n_1,\dots,n_t$ will be non-negative integers. Let

$$f(n_1,\dots,n_t) = a\sum_{i=1}^t \binom{n_i}{2} + b\sum_{i=1}^{t-1} n_i n_{i+1},$$

and let

$$f_c(n_1,\dots,n_t) = f(n_1,\dots,n_t) + b\, n_t n_1.$$

We call f the *cost* and f_c the *cyclic cost*. Sometimes it will be easier to work with the cyclic cost f_c because of the extra symmetry involved, and then to deduce results about the 'real' cost f.

We need to consider two cases, when $a \le 2b$ and when $a \ge 2b$. In the former case the 'internal' edges (corresponding to the terms $\binom{n_i}{2}$) are cheap relative to the 'cross' edges (corresponding to the terms $n_i n_{i+1}$), and in the latter case this is reversed. In the former case we need to determine the minimum cost: this is done in Lemma 4.1 below, which will be used to establish (11) in Section 5. In the latter case we need to determine both minimum and the maximum costs: the minimum cost is given in Lemma 4.2, which will be used to prove (14) in Section 6; and the maximum cost is given in Lemma 4.3, which will be used to prove Lemma 8.1 in Section 8.

Lemma 4.1. *Let $0 < a \le 2b$, let t be odd, and suppose that $\sum_{i=1}^{t} n_i = s(t+1)/2$ for a positive integer s. Then for the t-vector $(s,0,s,\dots,0,s)$ we have*

$$f(n_1,\dots,n_t) \ge f(s,0,s,\dots,0,s) = a\,\frac{t+1}{2}\binom{s}{2}.$$

Lemma 4.2. *Let $a \ge 2b > 0$, let $t \ge 3$, and suppose that $\sum_{i=1}^{t} n_i = st$ for a positive integer s. Then for the t-vector $(s,s,\dots,s)$ we have*

$$f_c(n_1,\dots,n_t) \ge f_c(s,s,\dots,s) = t\left(a\binom{s}{2} + bs^2\right).$$

Lemma 4.3. *Let $a \ge 2b > 0$, and let $t \ge 3$. Let s be a positive integer, and let $0 \le n_1,\dots,n_t \le s$. Let $l = \sum_{i=1}^{t} n_i$, and write l as $qs + r$ where $0 \le r < s$. Let $\tilde{n}_i = s$ for $i = 1,\dots,q$ let $\tilde{n}_{q+1} = r$ and let $\tilde{n}_i = 0$ for $q+1 \le i \le t$. [Thus we may picture $(\tilde{n}_1,\dots,\tilde{n}_t)$ as $(s,s,\dots,s,r,0,\dots,0)$.] Then*

$$f_c(n_1,\dots,n_t) \le f_c(\tilde{n}_1,\dots,\tilde{n}_t).$$

[In fact here it suffices to have $a \ge b > 0$.]

The rest of this section is devoted to proving these three lemmas, and is not needed elsewhere in the paper. Readers may prefer to move on to the next section! We start with two observations, and then two lemmas which do not depend on the relative sizes of a and b.

Consider the t-vector $\mathbf{n} = (n_1,\dots,n_t)$. Let $n_0 = n_{t+1} = 0$. If $1 \le i < j \le t$, and we reverse the segment from n_{i+1} to n_{j-1}, then the change in the cost f or f_c of the t-vector is

$$b(n_i - n_j)(n_{j-1} - n_{i+1}). \qquad (7)$$

If we move 1 from n_3 to n_2 say, the change in the cost f or f_c is

$$a(n_2 - n_3 + 1) + b(n_1 + n_3 - 1 - n_2 - n_4). \tag{8}$$

The next lemma shows in particular (assuming $t \geq 4$) that there is a vector $\tilde{\mathbf{n}} = (\tilde{n}_1, \ldots, \tilde{n}_t)$ minimising $f(\mathbf{n})$, such that $\tilde{n}_1$ and $\tilde{n}_t$ are the two largest co-ordinate values, and $\tilde{n}_2$ and $\tilde{n}_{t-1}$ are the two smallest: to see this, first take J as $\{1, \ldots, t\}$ and then as $\{2, \ldots, t-1\}$.

Lemma 4.4. *Consider any t-vector $\mathbf{n} = (n_1, \ldots, n_t)$. Let $n_0 = n_{t+1} = 0$. Let $0 \leq i < k \leq t+1$, and consider the interval J from $i+1$ to $k-1$. If $n_i, n_k \leq n_j$ for each $j \in J$, then by reordering some of the co-ordinates n_j within J we may obtain a vector $\tilde{\mathbf{n}} = (\tilde{n}_1, \ldots, \tilde{n}_t)$ such that $f(\tilde{\mathbf{n}}) \leq f(\mathbf{n})$ and $\tilde{n}_{i+1}, \tilde{n}_{k-1} \geq \tilde{n}_j$ for each $j \in \tilde{J} = J \setminus \{i+1, k-1\}$. Similarly, if $n_i, n_k \geq n_j$ for each $j \in J$ then we may insist that $\tilde{n}_{i+1}, \tilde{n}_{k-1} \leq \tilde{n}_j$ for each $j \in \tilde{J}$.*

Proof. Consider the case when $n_i, n_k \leq n_j$ for each $j \in J$: the other case is similar. We may assume that $|J| \geq 2$ and $n_i \leq n_k$. Let $j_1 \in J$ satisfy $n_{j_1} \geq n_j$ for each $j \in J$. If $j_1 \neq i+1$ then reverse the segment from n_{i+1} to n_{j_1}, forming $\mathbf{n}'$. By (7), the change in cost is $b(n_i - n_{j_1+1})(n_{j_1} - n_{i+1}) \leq 0$. Thus we have moved n_{j_1} next to n_i without increasing the cost.

Now let $J' = \{i+2, \ldots, k-1\}$, and let $j_2 \in J'$ satisfy $n'_{j_2} \geq n'_j$ for each $j \in J'$. If $j_2 \neq k-1$ then reverse the segment from n'_{j_2} to n'_{k-1}, and again the cost does not increase. ∎

Lemma 4.5. *Let t be odd. Let l be positive integer. Then there is a t-vector $\mathbf{n} = (n_1, \ldots, n_t)$ which minimises $f(\mathbf{n})$ subject to $\sum_i n_i = l$, and satisfies*

$$n_1 \geq n_t \quad \textit{and} \quad \sum_{i>1,\,\mathrm{odd}} n_i \geq \sum_{i\ \mathrm{even}} n_i. \tag{9}$$

Proof. Let $\mathbf{n} = (n_1, \ldots, n_t)$ be any t-vector which minimises $f(\mathbf{n})$ subject to $\sum_i n_i = l$. By applying Lemma 4.4 first with $J = \{1, \ldots, t\}$, then with $J = \{2, \ldots, t-1\}$ and so on, we see that we may rearrange the co-ordinates to form a new vector $\mathbf{n}' = (n'_1, \ldots, n'_t)$ which also minimises $f(\mathbf{n})$ subject to $\sum_i n_i = l$, and further is such that each odd co-ordinate is at least each even co-ordinate; and by reversing the vector if necessary we can also insist that $n'_1 \geq n'_t$. Then (9) must hold. ∎

Now we consider the case when $a \leq 2b$, and consider a vector $\mathbf{n}$ as in the last lemma.

Lemma 4.6. *Let $0 < a \leq 2b$. Suppose that t is odd and the t-vector $\mathbf{n} = (n_1, \ldots, n_t)$ satisfies $n_1 \geq n_t$ and $\sum_{i>1,\,\mathrm{odd}} n_i \geq \sum_{i\,\mathrm{even}} (n_i - 1)$. Suppose also* *that $n_i > 0$ for each even i with $1 < i < t$. Define the t-vector $\mathbf{n}'$ as follows: let $n'_1 = n_1$, and for $i = 2, \ldots, t$ let $n'_i = n_i - 1$ if i is even, and let $n'_i = n_i + 1$ if i is odd. Then $f(\mathbf{n}') \le f(\mathbf{n})$.*

Proof.

$$\begin{aligned} f(\mathbf{n}') - f(\mathbf{n}) = a & \left(- \sum_{i \text{ even}} (n_i - 1) + \sum_{i>1 \text{ odd}} n_i \right) \\ & + b \left(-(n_1 - n_t) - 2 \sum_{i>1 \text{ odd}} n_i - (n_2 - 1) + 2 \sum_{i \text{ even}} (n_i - 1) \right) \\ & \le (a - 2b) \left(\sum_{i>1 \text{ odd}} n_i - \sum_{i \text{ even}} (n_i - 1) \right) \le 0. \end{aligned}$$

∎

It is convenient to prove a strengthened version of Lemma 4.1, which yields that result immediately.

Lemma 4.7. *Let $0 < a \le 2b$, let t be odd, and suppose that $\sum_{i=1}^{t} n_i = l$. For $i = 1, \ldots, t$ let n_i^* be 0 for i even, and for i odd let n_i^* be $\lfloor \frac{2l}{t+1} \rfloor$ or $\lceil \frac{2l}{t+1} \rceil$ and such that $\sum_{i=1}^{t} n_i^* = l$. Then*

$$f(n_1, \ldots, n_t) \ge f(n_1^*, \ldots, n_t^*).$$

Proof. We prove that the lemma holds for all odd $t \ge 1$ by induction on t. The case $t = 1$ is trivial. Let $t = 2k+1$ for some integer $k \ge 1$, and suppose that the result holds for $t-2$. Let ν^* denote the minimum value of $f(\mathbf{n})$ subject to $\sum_{i=1}^{t} n_i = l$. By Lemmas 4.5 and 4.6, there exists a vector achieving this value which also satisfies $n_i = 0$ for some even i with $1 < i < t$. Hence by Lemma 4.4 there is such a vector $\tilde{\mathbf{n}}$, such that also $\tilde{n}_1$ and $\tilde{n}_t$ are the two largest co-ordinate values and $\tilde{n}_2$ or $\tilde{n}_{t-1}$ is the smallest co-ordinate value 0, and so by reversing the vector if necessary we have $\tilde{n}_{t-1} = 0$. (We care here only about $\tilde{n}_{t-1}$.) Then

$$\begin{aligned} \nu^* &= f(\tilde{\mathbf{n}}) = f(\tilde{n}_1, \ldots, \tilde{n}_{t-2}) + a \binom{\tilde{n}_t}{2} \\ &\ge a \sum_{j=1}^{k} \binom{m_j}{2} + a \binom{\tilde{n}_t}{2} \end{aligned}$$

for some $m_1, \ldots, m_k$ with $\sum_{j=1}^{k} m_j = l - \tilde{n}_t$, by the induction hypothesis. But it is easy to show that this last quantity is at least $f(\mathbf{n}^*)$, where $\mathbf{n}^*$ is as in the lemma, by the convexity of the function $x(x-1)/2$. ∎

We now consider the case when $a \geq 2b$. We first prove Lemma 4.2 concerning the minimum cost, and then Lemma 4.3 concerning the maximum cost.

Proof of Lemma 4.2. Recall that $a \geq 2b > 0$, and $t \geq 3$. Let $\mathbf{n} = (n_1, \ldots, n_t)$ minimise $f_c(\mathbf{n})$ subject to $\sum_i n_i = ts$, and further minimise $\sum_i n_i^2$. We shall show that each $n_i = s$, which will prove the lemma.

Let x be the maximum of the values $n_1, \ldots, n_t$ and let y be the minimum value. Assume for a contradiction that $y < x$. Then in fact $y \leq x - 2$ since $\sum_i n_i$ is divisible by t. Suppose without loss of generality that $n_2 = x$, and n_1 or n_3 is $< x$.

Suppose first that $n_1 \geq n_3$. If we move 1 from n_2 to n_3 then as in (8), the change in the cost f_c is

$$-(a - 2b)(n_2 - n_3 - 1) - b(n_1 + n_2 - n_3 - n_4 - 1).$$

Since the new vector has a smaller sum of squares, and since $n_2 - n_3 - 1 \geq 0$, $n_1 \geq n_3$ and $n_2 \geq n_4$, it follows from our choice of $\mathbf{n}$ that $n_1 = n_3$ and $n_2 = n_4 = x$. Similarly if we assume that $n_3 \geq n_1$ we find that $n_1 = n_3$. Hence without any assumption on how n_1 compares with n_3, we know that $n_1 = n_3$ and $n_2 = n_4 = x$.

Repeating this argument, we see that t must be even, $n_1 = n_3 = \cdots = n_{t-1} = y$ and $n_2 = n_4 = \cdots = n_t = x$. But now if replace each x by $x - 1$ and each y by $y + 1$, we strictly decrease the sum of squares, and change the cost f_c by

$$a(t/2)(y - x + 1) + bt((x - 1)(y + 1) - xy) = -(t/2)(a - 2b)(x - y - 1) \leq 0;$$

and this contradicts our choice of $\mathbf{n}$. ∎

Proof of Lemma 4.3. Suppose that $\mathbf{n} = (n_1, \ldots, n_t)$ maximises $f_c(\mathbf{n})$ subject to $0 \leq n_i \leq s$ for each i and $\sum_i n_i = l$. We make several easy deductions.

(a) The positive values are consecutive; for otherwise the cost would increase if we moved two blocks of positive values together.

(b) There cannot be adjacent values n_i in $\{1, \ldots, s-1\}$. For suppose that $0 < n_2, n_3 < s$. By symmetry we may assume that $n_1 \geq n_4$ (where $n_1 \equiv n_4$ if $t = 3$). Hence we may assume that $n_2 \geq n_3$; for if not, and we swap n_2 and n_3, then as in (7) the change in f_c is $b(n_1 - n_4)(n_3 - n_2) \geq 0$. But now if we move 1 from n_3 to n_2, the change in f_c is $(a - b)(n_2 - n_3 + 1) + b(n_1 - n_4) > 0$.

If the sum l is at most s then by (a) and (b), some $n_i = l$ and we are done. So we may assume that $l > s$.

(c) The values equal to s are consecutive. For otherwise, there would be values s, x, s, y in cyclic order, where $x, y < s$, the pair s, x is consecutive, and the pair s,y is consecutive. But then we could reverse the 'inner' segment from x to s, and as in (7) the change in f_c would be $b(s-x)(s-y)>0$.

If no value n_i is 0 then at most one value is less than s, and we are done; so we may suppose that some value is 0. It remains only to exclude the pattern $0,x,s,\ldots,s,y,0$, where $0<x\leq y<s$ (and where the first and last 0 may correspond to the same co-ordinate, or they may correspond to a string of 0's). But if we move 1 from x to y the change in f_c is $a(y-x+1)>0$, and so this pattern does not occur. ∎

5. The 'few short edges' case

In this section we prove the result (3) in the theorem. Thus we assume here that $p_1 \leq p_2(1-p_2)$, and so $a \leq 2b$ in the notation of Section 3. By the deterministic inequality (1), we need only prove a lower bound for the span.

Recall the standard lower bound on the chromatic number $\chi(G)$, that $\chi(G) \geq |V|/\alpha(G)$, as mentioned at the end of the first section. Given a network G,x let us call a subset U of the nodes *t-assignable* if there is a feasible assignment $\phi : U \to \{1,\ldots,t\}$; and let the *$t$-assignable number* $\alpha_t(G,x)$ be the maximum size $|U|$ of a t-assignable set U. We shall use the easy lower bound, extending the one above for the chromatic number, that

$$\text{span}(G,x) \geq |V|\ t/\alpha_t(G,x)\ -(t-1), \tag{10}$$

see for example [10]. To prove this, note that if ϕ is a feasible assignment using channels $1,\ldots,s$ then

$$|V| \leq \left\lceil \frac{s}{t} \right\rceil \alpha_t \leq \frac{s+t-1}{t}\alpha_t.$$

Let t be a fixed (large) odd integer, let s be a positive integer, let U be a set of $l=s(t+1)/2$ nodes in $G_{n,\mathbf{p}}$, and let $\phi{:}U \to \{1,\ldots,t\}$. By Lemma 4.1,

$$\Pr(\phi \text{ is feasible}) \leq p_0^{\frac{t+1}{2}\binom{s}{2}} = p_0^{l(s-1)/2}. \tag{11}$$

Hence

$$\begin{aligned}\Pr(\alpha_t(G_{n,\mathbf{p}}) \geq l) &\leq \binom{n}{l} t^l p_0^{l(s-1)/2}\\ &\leq \left(\frac{ne}{l}\ t\ p_0^{(s-1)/2}\right)^l\\ &= \exp l(\ln n - \ln l - (s/2)\ln(1/p_0) + O(1))\\ &\to 0 \quad \text{as } n \to \infty\end{aligned}$$

if $s \geq 2(\ln n)/\ln(1/p_0)$. Thus

$$\Pr(\alpha_t(G_{n,\mathbf{p}}) \geq (t+1)(\ln n)/\ln(1/p_0)) \to 0 \quad \text{as } n \to \infty.$$

Let $\varepsilon > 0$. Let t be sufficiently large that $t/(t+1) \geq 1-\varepsilon$. Then we see from the last result and (10) that

$$\Pr(\text{span}(G_{n,\mathbf{p}}) \geq (1-\varepsilon)n \ln(1/p_0)/(\ln n)) \to 1 \quad \text{as } n \to \infty.$$

This completes the proof of the result (3).

6. The 'few long edges' case

In this section we prove the result (4) in the theorem (except that we shall postpone the proof of Lemma 6.1 below to the next sections). Thus we assume here that

$$(12) \qquad p_1 \geq p_2(1-p_2) \qquad \text{and so} \qquad a \geq 2b$$

in the notation of Section 3. We will be able to prove the lower bound on the span quite easily, but we will have to work harder for the upper bound. Let

$$\beta = (1/2)\ln(1/p_0) + \ln(1/(1-p_2)).$$

Consider first the lower bound. We argue much as in the last section. Let t be a fixed (large) integer, let $n_1, \dots, n_t$ be non-negative integers, and assume that $\sum_{i=1}^{t} n_i = (t+1)s$. Let $(s, \dots, s)$ denote the $(t+1)$-vector of s's. By Lemma 4.2, for the $(t+1)$-vector $(s, s, \dots, s)$,

$$(13) \quad f(n_1, \dots, n_t) = f_c(n_1, \dots, n_t, 0) \geq f_c(s, \dots, s) = (t+1)\left(a\binom{s}{2} + bs^2\right).$$

Let U be a set of $l = (t+1)s$ nodes in $G_{n,\mathbf{p}}$, and let $\phi : U \to \{1, \dots, t\}$. Then by (13),

$$(14) \quad \Pr(\phi \text{ is feasible}) \leq p_0^{(t+1)\binom{s}{2}}(1-p_2)^{(t+1)s^2} = \left(p_0^{(s-1)/2}(1-p_2)^s\right)^l.$$

Hence

$$\begin{aligned}\Pr(\alpha_t(G_{n,\mathbf{p}}) \geq l) &\leq \binom{n}{l} t^l \left(p_0^{(s-1)/2}(1-p_2)^s\right)^l \\ &\leq \left(\frac{ne}{l}\, t\, p_0^{(s-1)/2}(1-p_2)^s\right)^l \\ &= \exp l\,(\ln n - \ln l - \beta s + O(1)) \\ &\to 0 \quad \text{as } n \to \infty\end{aligned}$$

if $s \geq (\ln n)/\beta$. Now we use the lower bound (10) as before. Hence, for any $\varepsilon > 0$, we see by taking t large enough that

$$\Pr(\text{span}(G_{n,\mathbf{p}}) \geq (1-\varepsilon)\beta n/\ln n) \to 1 \quad \text{as } n \to \infty.$$

This completes the lower bound part of the proof of (4).

Now we start to prove the upper bound part of (4). We follow the lines of the proof in [6] of the upper bound for $\chi(G_{n,p})$, which in turn is based on the treatment in [2]. Let $t \geq 3$ be an integer, which later we shall choose to be large (but fixed). We shall define a precise function $s^* = s^*(n)$ in (16) below, such that $s^*(n) = (1+o(1))\beta^{-1}\ln n$. The next lemma shows that $\alpha_t(G_{n,\mathbf{p}})$ is very unlikely to be 'too small': we postpone the proof to the next sections.

Lemma 6.1. *For each integer $t \geq 3$,*

$$\Pr(\alpha_t(G_{n,\mathbf{p}}) < ts^*(n)) \leq e^{-n^{4/3+o(1)}}.$$

Let $k = k(n) = \lceil n/\ln^2 n \rceil$. Let us say that a network has property Q_n^t if there are n nodes, and for all subsets W of at least k nodes, the corresponding induced subnetwork has $\alpha_t \geq ts^*(k(n))$. The next two lemmas establish the upper bound part of (4).

Lemma 6.2. *For each integer $t \geq 3$,*

$$\Pr(G_{n,\mathbf{p}} \text{ has property } Q_n^t) \to 1 \quad \text{as } n \to \infty.$$

Proof. By Lemma 6.1,

$$\Pr(G_{n,\mathbf{p}} \text{ does not have property } Q_n^t) \leq 2^n e^{-k^{4/3+o(1)}} \to 0 \quad \text{as } n \to \infty. \qquad \blacksquare$$

Lemma 6.3. *Let $t \geq 3$, and consider deterministic networks G_n, x_n with property Q_n^t for $n = 1, 2, \ldots$. Then*

$$\text{span}(G_n, x_n) \leq \left(1 + \frac{2}{t}\right)\beta \frac{n}{\ln n}$$

for all sufficiently large n.

Proof. Consider a network G_n, x_n. Find a maximum sized t-assignable set U_1 and a feasible assignment $\phi_1 : U_1 \to \{1, \ldots, t\}$. Skip channel $(t+1)$, and delete the nodes in U_1. In the remaining network, find a maximum sized t-assignable set U_2 and a feasible assignment $\phi_2 : U_2 \to \{t+2, \ldots, 2t+1\}$. Skip channel $2(t+1)$, and delete the nodes in U_2. Continue this procedure until after deleting the set U_j say, we find that fewer than k nodes remain. Note that $jts^*(k) \leq n$.

The partial assignments $\phi_1, \phi_2, \ldots$ fit together to give a feasible assignment ϕ for $U_1 \cup \cdots \cup U_j$ using channels $1, \ldots, (t+1)j-1$. We may extend ϕ to a feasible assignment for the whole network with span at most

$$(t+1)j + 2k \leq (t+1)\frac{n}{ts^*(k)} + 2k = (1+o(1))\frac{t+1}{t}\beta\frac{n}{\ln n}. \qquad \blacksquare$$

7. Lower bound for α_t

It remains only to prove Lemma 6.1. We start the proof in this section. The tedious second moment calculations are given in the following section. Recall that we are assuming that (12) holds, and that t is a fixed integer at least 3.

Consider a network G, x consisting of a graph G with set $V = \{v_1, \ldots, v_n\}$ of n nodes, and with edge lengths x_e equal to 0, 1 or 2. For $U \subseteq V$ call an assignment $\phi : U \to \{1, \ldots, t\}$ *cyclically feasible* for U if it is feasible and further there is no edge of length 2 between a node assigned channel 1 and a node assigned channel t. For $U \subseteq V$ with $|U| = ts$ the *canonical t-assignment* assigns channel 1 to the s nodes v_i with least index i, channel 2 to the next s nodes, and so on. A set U of ts nodes is (t,s)-*good* if the canonical t-assignment is cyclically feasible.

Let $f(G,x)$ be the maximum number of sets in a collection of $(t, s^*(n))$-good sets $S_1, S_2, \ldots$ such that $|S_i \cap S_j| \leq 1$ whenever $i \neq j$. Let $Y_n = f(G_{n,\mathbf{p}})$ and let $\mu_n = \mathrm{E}(Y_n)$. We shall show that

$$\mu_n \geq n^{5/3+o(1)}. \tag{15}$$

Note that if we change the length of any one edge, this can change $f(G,x)$ by at most 1. Hence by the bounded differences inequality, see for example [7, 8],

$$\Pr(Y_n - \mu_n \leq y) \leq \exp(-4y^2/n^2)$$

for any $y > 0$. In particular,

$$\begin{aligned}
\Pr(\alpha_t(G_{n,\mathbf{p}}) < ts^*(n)) &= \Pr(Y_n = 0) \\
&\leq \Pr(Y_n - \mu_n \leq \mu_n) \\
&\leq \exp(-4\mu_n^2/n^2) \\
&\leq e^{-n^{4/3+o(1)}}.
\end{aligned}$$

It remains then to prove (15) in order to complete the proof of Lemma 6.1.

Given a network G, x with nodes $v_1, \ldots, v_n$, for each $1 \leq n' \leq n$ we define $g(G, x, n')$ as follows. Let $\mathcal{S}$ denote the collection of all $(t, s^*(n))$-good sets $S \subseteq \{v_1, \ldots, v_{n'}\}$: then $g(G, x, n')$ is the number of sets $S \in \mathcal{S}$ such that

$|S \cap S'| \leq 1$ for each distinct set $S' \in \mathcal{S}$. (We restrict our attention to $v_1, \dots, v_{n'}$ in order to have just enough $(t, s^*(n))$-good sets, so that there will be many 'lonely' ones, nearly disjoint from all others.) Of course $f(G,x) \geq g(G,x,n')$. Thus the following lemma will prove (15) and so complete the story.

Lemma 7.1. *For all n sufficiently large, for a suitable choice of n'*

$$\mathrm{E}[g(G_{n,\mathbf{p}}, n')] \geq n^{5/3+o(1)}.$$

The proof of Lemma 7.1 will follow the standard second moment route, as for example in [6], though with some complications. It is described in the next section.

8. A second moment calculation

For any positive integers n, t and s let $E(n,t,s)$ denote the expected number of (t,s)-good sets in $G_{n,\mathbf{p}}$. Then

$$E(n,t,s) = \binom{n}{ts} p_0^{t\binom{s}{2}} (1-p_2)^{ts^2}.$$

For any positive integers n and t and real $s > 0$ with $st < n$, let

$$\hat{E}(n,t,s) = (2\pi)^{-\frac{1}{2}} n^{n+\frac{1}{2}} (n-ts)^{-(n-ts+\frac{1}{2})} (ts)^{-(ts+\frac{1}{2})} p_0^{t\binom{s}{2}} (1-p_2)^{ts^2}.$$

Then by Stirling's formula, much as in [6], $E(n,t,s) = (1+o(1))\hat{E}(n,t,s)$ if t and $s = s(n)$ are positive integers and both ts and $n - ts \to \infty$ as $n \to \infty$.

Now, as before, let t be a fixed positive integer. Let $s_1 = s_1(n) > 0$ satisfy

$$s_1 = \beta^{-1}(\ln n - \ln\ln n) + o(\ln\ln n).$$

Then

$$\ln \hat{E}(n,t,s_1) = ts_1(\ln n - \ln\ln n + \gamma - \beta s_1) + O(\ln\ln n),$$

where the constant γ is given by

$$\gamma = 1 - \ln t + \ln \beta - \frac{1}{2}\ln p_0.$$

Now let $s_2 = s_2(n)$ be given by

$$s_2 = s_2(n) = \beta^{-1}(\ln n - \ln\ln n + \gamma).$$

Then for $|x|=O(1)$, we have $\ln \hat{E}(n,t,s_2+x)=-tx\ln n+O(\ln\ln n)$. We set $x=x(n)=-\frac{5}{3t}$, and let $s^*=s^*(n)=\lfloor s_2+x\rfloor$, that is

$$s^*(n) = \left\lfloor \beta^{-1}(\ln n - \ln\ln n + \gamma) - \frac{5}{3t} \right\rfloor . \tag{16}$$

Then

$$E(n',t,s^*(n)) = n^{5/3+o(1/\ln\ln n)}$$

for some integer $n'=n'(n)\le n$ with $n'=\Omega(n)$, arguing much as in [6]. Thus it suffices to prove that

$$\mathrm{E}[g(G_{n,\mathbf{p}},n')] \ge (1+o(1))E(n',t,s(n)). \tag{17}$$

Let us write simply s for $s^*(n)$ from now on. Given a set S of ts nodes in $G_{n,\mathbf{p}}$, let F_S denote the event that S is (t,s)-good. (The 'F' is for **f**easible.) Also, for $l=0,\dots,ts$ let $Z_l(S)$ be the number of (t,s)-good sets $S'\subseteq\{v_1,\dots,v_{n'}\}$ with $|S\cap S'|=l$. Let $Z(S)=\sum_{l=2}^{ts-1} Z_l(S)$. The sums and the maximum indicated as over S below are over all $S\subseteq\{v_1,\dots,v_{n'}\}$ with $|S|=ts$. We have

$$\begin{aligned}\mathrm{E}(g(G_{n,\mathbf{p}},n')) &= \sum_S \Pr(F_S)\Pr(Z(S)=0|F_S)\\ &\ge \sum_S \Pr(F_S)(1-\mathrm{E}(Z(S)|F_S))\\ &\ge E(n',t,s)(1-\max_S \mathrm{E}(Z(S)|F_S)).\end{aligned}$$

To prove (17), it will suffice to show that

$$\max_S \mathrm{E}(Z(S)|F_S) = o(1).$$

For $l=2,\dots,ts-1$ let

$$F_l = \max_S \mathrm{E}(Z_l(S)|F_S).$$

We shall in fact show that

$$\sum_{l=2}^{ts-1} F_l = o(1), \tag{18}$$

which will complete the proof.

The idea from now on is as follows. The next lemma will give a natural upper bound $\tilde{F}_l$ for F_l. We then show that $\tilde{F}_2$ is quite small, that each $\tilde{F}_{qs}$ is very small for $q=1,\dots,t-1$ and that $\tilde{F}_{ts-1}$ is small. Other values $\tilde{F}_l$ are compared to these values, and we can deduce (18). We need to consider three ranges of values for l.

Lemma 8.1. *Let A and B be sets of ts nodes in $G_{n,\mathbf{p}}$. Suppose that $|A \cap B| = l$, where $2 \le l \le ts-1$. Write l as $qs+r$ where $0 \le r < s$. (Thus $0 \le q \le t-1$.) Let $x = q\binom{s}{2} + \binom{r}{2}$. Let $y=0$ if $q=0$, let $y=(q-1)s^2+rs$ if $1 \le q \le t-2$, and let $y=(q-1)s^2+2rs$ if $q=t-1$. Then*

$$\Pr(F_A|F_B) \le \Pr(F_A) p_0^{-x}(1-p_2)^{-y}.$$

Proof. Let $A = A_1 \cup \cdots \cup A_t$ be the canonical partition of A. Let $n_i = |A_i \cap B|$. Let $x' = \sum_{i=1}^{t} \binom{n_i}{2}$, and let $y' = \sum_{i=1}^{t-1} n_i n_{i+1} + n_1 n_t$. Then

$$\Pr(F_A|F_B) \le \Pr(F_A) p_0^{-x'}(1-p_2)^{-y'} = \Pr(F_A) e^{ax'+by'}.$$

But by the assumption (12) and Lemma 4.3, we have $ax'+by' \le ax+by$. Hence

$$e^{ax'+by'} \le e^{ax+by} = p_0^{-x}(1-p_2)^{-y},$$

which completes the proof. ∎

Now we consider the three ranges for l.

(a) Let us first consider small values of l. Let $2 \le l \le s$. Then by Lemma 8.1

$$F_l \le \tilde{F}_l = \binom{ts}{l}\binom{n'-ts}{ts-l} \Pr(F_A) p_0^{-\binom{l}{2}}.$$

Thus for $\tilde{F}_2$ in particular we have

$$\begin{aligned} \tilde{F}_2 &\le \binom{ts}{2}\binom{n'-ts}{ts-2} \Pr(F_A) p_0^{-1} \\ &= O(s^4/n^2)\binom{n'}{ts} \Pr(F_A) \\ &= O(\ln^4 n/n^2) E(n',t,s(n)) \\ &= n^{-\frac{1}{3}+o(1)}. \end{aligned}$$

Also, for each $2 \le l \le s$ we have

$$\begin{aligned} \frac{\tilde{F}_l}{\tilde{F}_2} &= \frac{\binom{ts}{l}\binom{n'-ts}{ts-l}}{\binom{ts}{2}\binom{n'-ts}{ts-2}} p_0^{1-\binom{l}{2}} \\ &= \frac{2((ts-2)_{(l-2)})^2}{l!(n'-2ts+l)_{(l-2)}} p_0^{1-\binom{l}{2}} \\ &\le \left(\frac{(ts)^2}{(n'-2ts)} p_0^{-\frac{l+1}{2}} \right)^{l-2}. \end{aligned}$$

But

$$p_0^{-\frac{l+1}{2}} \leq p_0^{-\frac{s+1}{2}} = e^{a(s+1)/2} = e^{(\frac{a}{a+2b}+o(1))\ln n} = n^{\frac{a}{a+2b}+o(1)},$$

and so the term in the large brackets above is

$$O\left(\frac{\ln^2 n}{n} n^{\frac{a}{a+2b}+o(1)}\right) = O\left(n^{-\frac{2b}{a+2b}+o(1)}\right).$$

Hence

$$\frac{\tilde{F}_l}{\tilde{F}_2} = O\left(n^{-(\frac{2b}{a+2b}+o(1))(l-2)}\right).$$

It follows that

$$\sum_{l=2}^{s} \tilde{F}_l = O(\tilde{F}_2) \leq n^{-\frac{1}{3}+o(1)},$$

and

$$\tilde{F}_s = e^{-\Omega(\ln^2 n)}$$

(which is very small).

(b) Next we consider intermediate values of l. Let $l=qs+r$ where $1 \leq q \leq t-2$ and $0 \leq r \leq s$. By Lemma 8.1

$$F_l \leq \tilde{F}_l = \binom{ts}{l}\binom{n'-ts}{ts-l} \Pr(F_A) p_0^{-q\binom{s}{2}-\binom{r}{2}} (1-p_2)^{-(q-1)s^2-rs}.$$

Much as before we have

$$\begin{aligned}
\frac{\tilde{F}_l}{\tilde{F}_{qs}} &= \frac{\binom{ts}{l}\binom{n'-ts}{ts-l}}{\binom{ts}{qs}\binom{n'-ts}{ts-qs}} p_0^{-\binom{r}{2}} (1-p_2)^{-rs} \\
&= \frac{((ts-qs)_{(r)})^2}{l_{(r)}(n'-2ts+l)_{(r)}} p_0^{-\binom{r}{2}} (1-p_2)^{-rs} \\
&\leq \left(\frac{(ts-qs)^2}{(qs)(n'-2ts)} p_0^{-\frac{r-1}{2}} (1-p_2)^{-s}\right)^r.
\end{aligned}$$

But

$$\ln\left(p_0^{-s/2}(1-p_2)^{-s}\right) = \beta s = \ln n - \ln\ln n + O(1),$$

and so the term in the large brackets above is

$$O\left(\frac{\ln n}{n} p_0^{-s/2} (1-p_2)^{-s}\right) = O(1).$$

Hence

$$\ln\left(\frac{\tilde{F}_l}{\tilde{F}_{is}}\right) = O(\ln n).$$

Now let us focus on the terms $\tilde{F}_{is}$ for $i=1,\ldots,t-1$. We have

$$\ln(\tilde{F}_{is}) = \sum_{j=1}^{i-1} \ln\left(\frac{\tilde{F}_{(j+1)s}}{\tilde{F}_{js}}\right) + \ln(\tilde{F}_s) = O(\ln n) - \Omega(\ln^2 n) = -\Omega(\ln^2 n).$$

Hence for each $q=1,\ldots,t-2$

$$\sum_{l=qs}^{qs+s} \tilde{F}_l = e^{-\Omega(\ln^2 n)},$$

and so

$$\sum_{l=s}^{(t-1)s} \tilde{F}_l = e^{-\Omega(\ln^2 n)}.$$

(c) It remains to consider the large values of l. Let $l=(t-1)s+r$ where $1 \le r \le s-1$. By Lemma 8.1

$$F_l \le \tilde{F}_l = \binom{ts}{l}\binom{n'-ts}{ts-l} \Pr(F_A) p_0^{-(t-1)\binom{s}{2}-\binom{r}{2}} (1-p_2)^{-(t-2)s^2-2rs}.$$

In particular,

$$\tilde{F}_{ts-1} = (ts)(n'-ts)p_0^{s-1}(1-p_2)^{2s} = O(n \ln n\ e^{-2\beta s}) = O(n^{-1+o(1)}).$$

Let $1 \le i \le s-1$. Much as before we have

$$\begin{aligned}
\frac{\tilde{F}_{ts-i}}{\tilde{F}_{ts-1}} &= \frac{\binom{ts}{i}\binom{n'-ts}{i}}{(ts)(n'-ts)}\ p_0^{\binom{i}{2}+i(s-i)-s+1}(1-p_2)^{2is-2s} \\
&= \frac{((ts-1)_{(i-1)})^2}{i!}(n'-ts-l)_{(i-1)}\ p_0^{\frac{1}{2}(2s-i-2)(i-1)}(1-p_2)^{2s(i-1)} \\
&\le \left(tsnp_0^{(2s-i-2)/2}(1-p_2)^{2s}\right)^{i-1}.
\end{aligned}$$

But

$$\begin{aligned}
p_0^{(2s-i-2)/2}(1-p_2)^{2s} &\le p_0^{s/2}(1-p_2)^{2s} \\
&= \exp(-s(a/2\ + b)) \\
&= \exp(-(1+o(1))\ln n(1+b/\beta)) \\
&= n^{-(1+b/\beta+o(1))}.
\end{aligned}$$

Hence, for each $1 \le i \le s-1$,

$$\frac{\tilde{F}_{ts-i}}{\tilde{F}_{ts-1}} \le \left(n^{-b/\beta+o(1)}\right)^{i-1}.$$

Therefore,

$$\sum_{l=(t-1)s+1}^{ts-1} \tilde{F}_l = O(\tilde{F}_{ts-1}) = O\left(n^{-1+o(1)}\right).$$

Finally, we may put the three ranges for l together, to see that

$$\sum_{l=2}^{ts-1} F_l \le n^{-\frac{1}{3}+o(1)} = o(1),$$

which completes the proof of (18) and thus of the entire theorem.

9. Concluding remarks

Suppose that we think of our departure point as the familiar problem of analysing the chromatic number $\chi(G_{n,p})$ of a random graph with constant edge probability p. Then we have taken the natural first step in investigating the span of random channel assignment problems, by analysing the span for random networks where the edge-lengths are 0,1 or 2 and the corresponding probabilities are constants. We found that the behaviour of the span is more delicate than that of the chromatic number, in that there is a 'phase change' phenomenon.

It would be interesting to pursue these investigations further, for example to allow different edge-lengths, or to consider sparse random models where $p_0 = 1 - o(1)$, or to consider greedy assignment methods, or to introduce demands and co-site constraints.

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